Transfinite iteration functionals and ordinal arithmetic

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Abstract. Although transfinite iteration functionals have been used in the past to construct ever-larger initial segments of the ordinals ([5],[1]), there appears to be little investigation into the nature of the functionals themselves. In this note, we investigate the relationship between (countable) transfinite iteration and ordinal arithmetic. While there is a nice connection between finite iteration and addition, multiplication, and exponentiation, we show that this it is lost when passing to the transfinite and investigate a new equivalence relation on ordinal functionals with respect to which we restore it.

1. Introduction

The use of functionals of higher type for defining ever-increasing initial segments of (countable) ordinals is not a new idea—Feferman uses a notion of transfinite iteration functionals of finite type in [5] and Aczel extends this work to transfinite type in [1]. However, in none of this research does there appear to be an analysis of the iteration functionals themselves. Specifically, we wish to understand more completely the relationship between iteration and (ordinal) arithmetic. Furthermore, our original motivation for this investigation was an interest in understanding definability of ordinals when the tools for functional definition are restricted (this is the subject of the author's Ph.D. thesis [3]). A natural way to implement such restrictions is to use some version of a typed λ-calculus; doing so necessitates that our work must take place in a structure that can be used as a model for at least the simply typed λ-calculus.

We first consider finite iteration to determine just what such an analysis should yield. When we identify numbers with iterators (for example, by representing numbers as Church numerals in the λ-calculus), we make explicit the view that the functional equivalent of counting is iterated function application. Considering counting to be the basic operation in the universe of numbers, we are led to ask what the numeric analogue of the basic operation of functionality is under this equivalence. That basic operation is, of course, application. In other words, to what does the interaction between iteration and application correspond in the universe of numbers? The most elementary interaction consists of iterating a function, say m times, then iterating it again, say n times. The result, of course, is the same as iterating the function $m+n$ times. In other words, application at the object level corresponds to addition: writing I^σ_m for the type-σ m-fold iteration functional, we have $I^\sigma_n f(I^\sigma_m fx) = I^\sigma_{m+n} fx$ (associating application to the left). Since iteration is defined as a higher-type functional, two more kinds of application are basic: application at function level and application of one iteration functional to another. The results of such applications are easy to establish: $I^\sigma_n(I^\sigma_m f) = I^\sigma_{mn} f$ and $I^{\sigma\to\sigma}_n(I^\sigma_m) = I^\sigma_{m^n}$. Thus the fundamental operation of functionality translates back to the universe of numbers as fundamental operations of arithmetic: addition, multiplication, and exponentiation. By viewing countable ordinals as being obtained by transfinitely counting, the identification of numbers with iterated function application extends to identifying countable ordinals with transfinite iteration. As such, we expect to see the correspondence between application and ordinal addition, multiplication, and exponentiation extend to transfinite iteration, and the purpose of this note is to investigate in what way it does so.

As already mentioned, such functionals have been used in the past, most notably in connection with defining ever-larger initial segments of the constructive ordinals. However, such work has

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mostly focused on the definable ordinals, rather than the iteration functionals themselves. Moreover, the intuitive definition of $I_\omega f x$ as $\lim_{n\to\omega}\{I_n f x\}$ is not well-defined for all arguments f. In order to compensate for this, authors have usually taken the ω-iterate of a function f at x to be $\sup_{n\in\omega}\{I_n f x\}$. Although these definitions are equivalent for the functions used in practice to define ordinals (which are increasing), the supremum definition results in anomalies when the focus is on iteration of arbitrary functions. For example, if $f(x) = 0$ for all x, then $\sup_{n\in\omega}\{I_n f 1\} = 1$, whereas we would expect $I_\omega f 1$ to be 0.

Here, we define α-iterator functionals I^ρ_α for each finite type ρ by using the lim sup operator, thus staying as close as possible to the ideal of limit behavior while maintaining totality of the functionals. We show in Section 3 that if we restrict attention to monotone functions, iteration corresponds exactly to ordinal arithmetic, as we insist (Thm. 3.7). However, these results cannot be extended to non-monotone functions or higher type levels—for example, we define a type-2 monotone functional Φ such that in general, $I_\gamma(I_\alpha\Phi) \neq I_{\alpha\gamma}\Phi$. The crux of the difficulty (which also arises when supremum is used in the definition of I_ω) is that unless the limit of a sequence exists one cannot control the behavior of subsequences. We resolve this in Section 4 by introducing a new equivalence relation $=^{\rm hp}$ on ordinal functionals which allows us to focus our attention on arguments for which the appropriate limits do exist (although, as mentioned above, we do not eliminate such arguments from consideration altogether). We then establish the desired correspondence relative to $=^{\rm hp}$ for all functionals at all type levels (Thm. 4.8). As $=^{\rm hp}$ is just equality on the ordinals themselves, we can make use of the correspondence to define larger ordinals through application of iteration functionals.

2. Preliminaries

We will work in two finite type structures over Ω, where Ω is the first uncountable ordinal. We define the full type structure $\mathrm{Tp}(\Omega) = \{\Omega_\sigma\}_\sigma$ and the hereditarily monotone type structure $\mathrm{Tp}_{\rm mon}(\Omega) = \{\Omega^{\rm mon}_\sigma\}_\sigma$ as follows. $\Omega_o = \Omega^{\rm mon}_o = \Omega$, and the order in both cases is the usual order on the ordinals. If Ω_σ and Ω_τ have been defined, then

$$\begin{aligned}\Omega_{\sigma\to\tau} &= \{f \mid f : \Omega_\sigma \to \Omega_\tau\}\\ \Omega^{\rm mon}_{\sigma\to\tau} &= \{f \mid f : \Omega^{\rm mon}_\sigma \to \Omega^{\rm mon}_\tau \text{ is monotone}\}\end{aligned}$$

where we say that f is *monotone* provided that $f(x) \le f(y)$ whenever $x \le y$. The order is defined pointwise in both cases: $f \le g$ if for all $x \in \Omega_\sigma$ $(x \in \Omega^{\rm mon}_\sigma)$, $f(x) \le g(x)$.

The pointwise definition of the order on $\Omega_{\sigma\to\tau}$ yields a pointwise characterization of supremums and infimums over an arbitrary index set I:

$$\Big(\sup_{i\in I}\{f_i\}\Big)(x) = \sup_{i\in I}\{f_i(x)\} \qquad \Big(\inf_{i\in I}\{f_i\}\Big)(x) = \inf_{i\in I}\{f_i(x)\}$$

Proposition 2.1 *For each type σ, if $X \subseteq \Omega_\sigma$, then* $\inf X$ *exists; moreover, if X is countable, then* $\sup X$ *exists.*

Proof. Both claims are proved by induction on σ. The existence of $\sup X$ in the base case follows from the regularity of Ω and the induction step is trivial. ■

The following definitions of lim sup, lim inf and limit are taken from Birkhoff [2, §X.9], but we have restricted attention to the case in which the nets are based on countable ordinals.

Definition For each type σ and countable ordinal ζ, if $\{x_\xi\}_{\xi<\zeta} \subseteq \Omega_\sigma$, then

$$\limsup_{\xi\to\zeta}\{x_\xi\} =_{\rm df} \inf_{\gamma<\zeta}\Big\{\sup_{\gamma\le\xi<\zeta}\{x_\xi\}\Big\} \qquad \liminf_{\xi\to\zeta}\{x_\xi\} =_{\rm df} \sup_{\gamma<\zeta}\Big\{\inf_{\gamma\le\xi<\zeta}\{x_\xi\}\Big\}$$

If there is $x \in \Omega_\sigma$ such that $\limsup_{\xi\to\zeta}\{x_\xi\} = x = \liminf_{\xi\to\zeta}\{x_\xi\}$, then we say that $\lim_{\xi\to\zeta}\{x_\xi\}$ exists and is equal to x.

Both Aczel [1] and the author [3] have defined transfinite type structures. In both cases, limit level function spaces are defined as a product over the function spaces of lower level (in fact, Aczel defines successor levels in the same way for the sake of uniformity). By extending the order to such spaces coordinatewise, it is not difficult to extend the results in this paper to such type structures.
Definition For each type σ and countable ordinal α, the α-iteration functional of type σ is the functional $I^\sigma_\alpha : \Omega_{\sigma\to\sigma} \to \Omega_{\sigma\to\sigma}$ defined by

$$I^\sigma_0 fx = x \qquad I^\sigma_{\alpha+1} fx = f(I^\sigma_\alpha fx) \qquad I^\sigma_\mu fx = \limsup_{\xi\to\mu}\{I^\sigma_\xi fx\}$$

where application associates to the left. We usually drop the type subscript whenever it is clear from context or irrelevant.

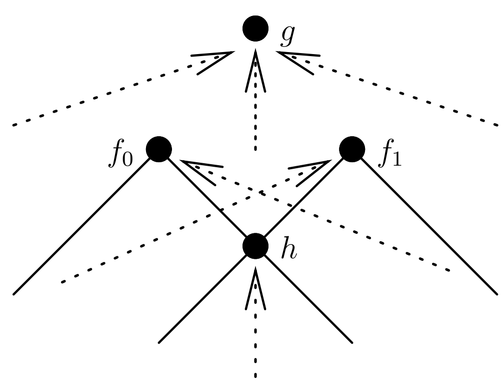
Note that by using the pointwise characterization of sup and inf, we can push arguments in and out of lim sup: $\limsup_{\xi\to\zeta}\{f_\xi x\} = (\limsup_{\xi\to\zeta}\{f_\xi\})(x)$. Applying this to the definition of I_μ for limit μ, we have $I_\mu fx = \limsup_{\xi\to\mu}\{I_\xi fx\} = (\limsup_{\xi\to\mu}\{I_\xi\})fx$, so $I_\mu = \limsup_{\xi\to\mu}\{I_\xi\}$.

We give two counterexamples to show that the correspondence between transfinite iteration and ordinal arithmetic need not hold. Let f be any ordinal function such that $f(2x) = 1$ and $f(2x+1) = 0$ when $x < \omega$. Then if $g =_{\mathrm{df}} I_2 f$, we have $g(2x) = 0$ and $g(2x+1) = 1$ for all $x < \omega$, so $I_\omega(I_2 f)(0) = I_\omega g0 = 0$. On the other hand, $I_{2\omega} f0 = I_\omega f0 = 1$. Of course, f is a rather poorly-behaved function, and one might hope that this difficulty would not arise for functions that are somehow well-behaved. For example, Aczel [1] restricts attention to hereditarily inflationary functionals. This is not an ideal resolution for us for two reasons: it requires a "pure" type structure (i.e., functionals always have the same domain and range) so that it makes sense to compare input and output, and such functionals do not yield a model in which the λ-calculus can be directly interpreted (since, e.g., constant functionals are λ-definable but not inflationary).

We give another example of the failure of application to correspond to arithmetic, this time using only monotone functionals. In particular, we cannot equate "well-behaved" with monotonicity. The type-2 functional to be iterated interchanges two functions. In this case, the double iterate will be the identity on either of those functions, and so the ω-iterate of the double iterate will also be the identity on either of the functions. However, the ω-iterate of the functional itself cannot be the identity, because it is alternating between the two. For the two functions, define $f_0(\alpha) = \alpha$, $f_1(\alpha) = 2$ for all α. Set $g =_{\mathrm{df}} \max\{f_0, f_1\}$ and $h =_{\mathrm{df}} \min\{f_0, f_1\}$, and define Φ by

$$\Phi(f) = \begin{cases} h, & f \le h \\ f_{1-i}, & f \le f_i, f \not\le h \\ g, & \text{otherwise} \end{cases}$$

Verifying that Φ is monotone is straightforward, though tedious; the following picture of the action of Φ should suffice:



We want to compare $I_\omega(I_2\Phi)f_0$ and $I_\omega\Phi f_0$. For the former, set $\Psi =_{\mathrm{df}} I_2\Phi$; then

$$\Psi(f) = \begin{cases} h, & f \le h \\ f_i, & f \le f_i, f \not\le h \\ g, & \text{otherwise} \end{cases}$$

In particular, $\Psi(f_i) = f_i$ for $i = 0, 1$, so $I_\omega(I_2\Phi)f_0 = I_\omega\Psi f_0 = f_0$. On the other hand, a direct computation shows that $I_\omega\Phi f_0 = g$, and therefore $I_\omega(I_2\Phi)f_0 \ne I_\omega\Phi f_0$.

What drives this example is the fact that f_0 and $\Phi(f_0)$ are not comparable—as a result, the sequence of iterates $\langle\Phi^n(f)\rangle_n$ does not have a limit, and therefore subsequences may have different limiting behavior than the sequence. We begin to repair the damage by analyzing iteration of monotone functionals which map each input to a comparable output. In this case the iterates form either non-decreasing or non-increasing sequences, and as a result subsequences will behave well. Unfortunately, the comparability requirement is too restrictive, because it is only guaranteed to hold when the order on the domain is total. Thus, it prevents us from establishing the connection between application of the iteration functionals and arithmetic at higher type. To push upwards, we develop the notion of hereditarily positive equality, with respect to which the correspondence is exact at all types.

3. Hereditarily Monotone Functionals

We partially investigated iteration functionals in $\mathrm{Tp}_{\mathrm{mon}}(\Omega)$ in [4]; the results here significantly extend this earlier work.

Lemma 3.1 *For each type σ and countable ordinal ζ, if $f_\xi \in \Omega^{\mathrm{mon}}_\sigma$ for all $\xi < \zeta$, then $\limsup_{\xi\to\zeta}\{f_\xi\} \in \Omega^{\mathrm{mon}}_\sigma$.*

Proof. The Lemma is proved by induction on σ. This is trivial if $\Omega^{\mathrm{mon}}_\sigma = \Omega^{\mathrm{mon}}$. Otherwise, suppose that $\sigma = \rho \to \tau$. If $x \in \Omega^{\mathrm{mon}}_\rho$, then $(\limsup\{f_\xi\})(x) = \limsup\{f_\xi x\} \in \Omega^{\mathrm{mon}}_\tau$ by induction, because each $f_\xi x \in \Omega^{\mathrm{mon}}_\tau$. Furthermore, if $x \le x'$ are elements of $\Omega^{\mathrm{mon}}_\rho$, then $(\limsup\{f_\xi\})(x) = \limsup\{f_\xi x\} \le \limsup\{f_\xi x'\} = (\limsup\{f_\xi\})(x)$, with the inequality holding because f_ξ is hereditarily monotone and $x \le x'$, so $f_\xi x \le f_\xi x'$ for all ξ. ■

Proposition 3.2 *For each type σ and countable ordinal α, I^σ_α is hereditarily monotone.*

Proof. The Proposition is proved by induction on α for all σ. If $\alpha = 0$, then I_α is just the functional that is constantly the identity on $\Omega^{\mathrm{mon}}_\sigma$, which is easily seen to be hereditarily monotone. Suppose that $\alpha = \gamma + 1$. First we must verify that I_α maps $\Omega^{\mathrm{mon}}_{\sigma\to\sigma}$ to itself. Suppose that $f \in \Omega^{\mathrm{mon}}_{\sigma\to\sigma}$ and $x \in \Omega^{\mathrm{mon}}_\sigma$. Then since $I_\gamma fx \in \Omega^{\mathrm{mon}}_\sigma$ by the induction hypothesis and f is hereditarily monotone by assumption, $I_\alpha fx = f(I_\gamma fx)$ is hereditarily monotone, and so $I_\alpha f \in \Omega^{\mathrm{mon}}_{\sigma\to\sigma}$. We must also verify that if x and x' are hereditarily monotone, $x \le x'$, then $I_\alpha fx \le I_\alpha fx'$, which is just as easy to do. Second, we must verify the monotonicity of I_α: if f, $f' \in \Omega^{\mathrm{mon}}_{\sigma\to\sigma}$ are such that $f \le f'$, then $I_\alpha f \le I_\alpha f'$. Fix any $x \in \Omega^{\mathrm{mon}}_\sigma$. Then $I_\alpha fx = f(I_\gamma fx) \le f(I_\gamma f'x) \le f'(I_\gamma f'x) = I_\alpha f'x$; the first inequality follows from the fact that $I_\gamma f \le I_\gamma f'$ (induction) and the second from the fact that $f \le f'$. This takes care of the successor case. If α is a limit, then $I_\alpha = \limsup_{\xi\to\alpha}\{I_\xi\}$ by definition; but this $\limsup$ is hereditarily monotone by induction and Lemma 3.1. ■

Definition We say that $\{f_\xi\}_{\xi<\zeta} \subseteq \Omega^{\mathrm{mon}}_\sigma$ is *non-decreasing* (*non-increasing*) if whenever $\alpha < \gamma < \zeta$, $f_\alpha \le f_\gamma$ ($f_\alpha \ge f_\gamma$, respectively). We use the same terminology when $\{f_\xi\}_{\xi<\zeta} \subseteq \Omega_\sigma$.

Lemma 3.3 *Fix any type σ and countable ordinal ζ, and let $\{f_\xi\}_{\xi<\zeta} \subseteq \Omega^{\mathrm{mon}}_\sigma$.*

1. *If the limit of a sequence exists, then it is the limit of any subsequence: if $\lim\{f_\xi\}$ exists, $q : \zeta' \to \zeta$ is non-decreasing, and $\lim_{\alpha\to\zeta'} q(\alpha) = \zeta$, then $\lim\{f_{q(\xi)}\}_{\xi<\zeta'}$ exists and is equal to $\lim\{f_\xi\}_{\xi<\zeta}$.*
2. *If $\{f_\xi\}$ is non-decreasing, then $\lim\{f_\xi\}$ exists and is equal to $\sup_{\xi<\zeta}\{f_\xi\}$.*

3. *If* $\{f_\xi\}$ *is non-increasing, then* $\lim\{f_\xi\}$ *exists and is equal to* $\inf_{\xi<\zeta}\{f_\xi\}$.

Proof. (1) In general, the lim inf of a subsequence is always greater than or equal to the lim inf of the sequence, and vice-versa for lim sup, so if $f = \lim_{\xi\to\zeta}\{f_\xi\}$, then $\liminf\{f_{q(\xi)}\} \geq f \geq \limsup\{f_{q(\xi)}\}$. But for any sequence $\{g_\mu\}_{\mu<\theta}$, $\liminf\{g_\mu\} \leq \limsup\{g_\mu\}$, so this implies that $\liminf f_{q(\xi)} = f = \limsup_{f_{q(\xi)}}$.

(2) Since $\{f_\xi\}$ is non-decreasing,

$$\liminf_{\xi\to\zeta}\{f_\xi\} = \sup_{\mu<\zeta}\big\{\inf_{\mu\leq\xi<\zeta}\{f_\xi\}\big\} = \sup_{\mu<\zeta}\{f_\mu\}$$

and

$$\limsup_{\xi\to\zeta}\{f_\xi\} = \inf_{\mu<\zeta}\big\{\sup_{\mu\leq\xi<\zeta}\{f_\xi\}\big\} = \sup_{0\leq\xi<\zeta}\{f_\xi\} = \liminf_{\xi\to\zeta}\{f_\xi\}.$$

(3) is similar to (2). ■

<u>Lemma 3.4</u> *Fix any types* σ *and* τ, $X \subseteq \Omega^{\mathrm{mon}}_\sigma$, *and let* $f : \Omega^{\mathrm{mon}}_\sigma \to \Omega^{\mathrm{mon}}_\tau$ *be monotone. If* $\sup X$ *exists, then* $f(\sup X) \geq \sup\{f(x) \mid x \in X\}$, *and if* $\inf X$ *exists, then* $f(\inf X) \leq \inf\{f(x) \mid x \in X\}$.

Proof. Both claims have similar proofs, so we just do the first. If $x \in X$, then $x \leq \sup X$, so by monotonicity of f, $f(x) \leq f(\sup X)$. Since x was chosen arbitrarily, $\sup\{f(x) \mid x \in X\} \leq f(\sup X)$. ■

<u>Lemma 3.5</u> *For each type* σ, *countable ordinal* ζ, $f \in \Omega^{\mathrm{mon}}_{\sigma\to\sigma}$, *and* $x \in \Omega^{\mathrm{mon}}_\sigma$:

1. *If* $I_{\zeta+1}fx \geq I_\zeta fx$, *then for all* $\gamma > \alpha \geq \zeta$, $I_\gamma fx \geq I_\alpha fx$.
2. *If* $I_{\zeta+1}fx \leq I_\zeta fx$, *then for all* $\gamma > \alpha \geq \zeta$, $I_\gamma fx \leq I_\alpha fx$.

Proof. Each clause is proved by a similar induction on γ; we do just the first. Throughout the proof, we make silent use of Lemma 3.3(1) to identify the limit of a sequence with the limit of a tail of that sequence, provided the former exists. If $\gamma = 0$, the claim is vacuous. Suppose that $\gamma = \delta+1$; by induction, it suffices to show that $I_\delta fx \leq I_\gamma fx$, and we do this by induction on δ. If $\delta = \alpha$, then this is just the hypothesis that $I_\zeta fx \leq I_{\zeta+1}fx$. The successor case is straightforward. Suppose that δ is a limit. By the main induction hypothesis, $\{I_\xi fx\}_{\zeta\leq\xi<\delta}$ is a non-decreasing sequence, so $I_\delta fx = \lim_{\xi\to\delta}\{I_\xi fx\} = \sup_{\zeta\leq\xi<\delta}\{I_\xi fx\}$ by Lemma 3.3(2). Now applying Lemma 3.4,

$$I_\gamma fx = f(I_\delta fx) = f\big(\sup_{\zeta\leq\xi<\delta}\{I_\xi fx\}\big) \geq \sup_{\zeta\leq\xi<\delta}\{f(I_\xi fx)\} = \sup_{\zeta\leq\xi<\delta}\{I_{\xi+1}fx\}.$$

This last sequence is a subsequence of $\{I_\xi fx\}_{\zeta\leq\xi<\delta}$, so it is non-decreasing, and therefore by Lemma 3.3(2) its supremum is a limit, and by Lemma 3.3(1) the limit is the same as that of the original sequence: $\sup_{\zeta\leq\xi<\delta}\{I_{\xi+1}fx\} = \lim\{I_{\xi+1}fx\} = \lim\{I_\xi fx\} = I_\delta fx$. So $I_\gamma fx \geq I_\delta fx$. This completes the induction step for successor γ. Finally, suppose that γ is a limit. Then by induction $\{I_\xi fx\}_{\xi<\gamma}$ is non-decreasing, so for any $\alpha < \gamma$, $I_\alpha fx \leq \sup_{\xi<\gamma}\{I_\xi fx\} = I_\gamma fx$. ■

<u>Proposition 3.6</u> *If* $f : \Omega^{\mathrm{mon}} \to \Omega^{\mathrm{mon}}$ *is a monotone function and* α *is a countable limit ordinal, then for all* β, $I_\alpha f\beta = \lim_{\xi\to\alpha} I_\xi f\beta$.

Proof. This follows from Lemmas 3.3 and 3.5 (taking $\zeta = 0$), because the order on Ω^{mon} is total. ■

We can now establish the connection between arithmetic of ordinals and application of iteration functionals at base type:

<u>Theorem 3.7</u> (Iteration Functionals in $\mathrm{Tp}_{\mathrm{mon}}(\Omega^{\mathrm{mon}})$) *Suppose* $f : \Omega^{\mathrm{mon}} \to \Omega^{\mathrm{mon}}$ *is a monotone function. Then for any* α *and* γ:

1. $I^\Omega_\alpha f \circ I^\Omega_\gamma f = I^\Omega_{\gamma+\alpha} f$.
2. $I^\Omega_\alpha(I^\Omega_\gamma f) = I^\Omega_{\gamma\alpha} f$.
3. $I^{\Omega\to\Omega}_\alpha(I^\Omega_\gamma) f = I^\Omega_{\gamma^\alpha} f$.

Proof. All three clauses are proved by induction on α; we do (2) as an example. Fix any ordinal β. If $\alpha = 0$, then $I_\alpha(I_\gamma f)\beta = \beta = I_{\alpha\gamma} f\beta$.

If $\alpha = \delta + 1$, then $I_\alpha(I_\gamma f)\beta = I_\gamma f(I_\delta(I_\gamma f)\beta) = I_\gamma f(I_{\gamma\delta} f\beta) = I_{\gamma\delta+\gamma} f\beta = I_{\gamma\alpha} f\beta$, where the second equality follows from the induction hypothesis and the third from part (1).

Suppose that α is a limit. By Prop. 3.6, $I_{\gamma\alpha} f\beta = \lim_{\xi\to\gamma\alpha}\{I_\xi f\beta\}$. Since $\{I_{\gamma\xi} f\beta\}_{\xi<\alpha}$ is a subsequence of $\{I_\xi f\beta\}_{\xi<\gamma\alpha}$ and the limit of the latter sequence exists,

$$\begin{aligned}
I_{\gamma\alpha} f\beta &= \lim_{\xi\to\gamma\alpha}\{I_\xi f\beta\} && \text{(Prop. 3.6)}\\
&= \lim_{\xi\to\alpha}\{I_{\gamma\xi} f\beta\} && \text{(Lemma 3.3(1))}\\
&= \limsup_{\xi\to\alpha}\{I_{\gamma\xi} f\beta\} && \text{(Definition of lim)}\\
&= \limsup_{\xi\to\alpha}\{I_\xi(I_\gamma f)\beta\} && \text{(Induction Hypothesis)}\\
&= \Big(\limsup_{\xi\to\alpha}\{I_\xi\}\Big)(I_\gamma f)(\beta) && \text{(Definition of lim sup)}\\
&= I_\alpha(I_\gamma f)\beta && \text{(Definition of } I_\alpha)
\end{aligned}$$

completing the proof. ∎

We show by example that the hypothesis of Lemma 3.5 need not be satisfied at higher type. It suffices to find a monotone function f such that $I_\gamma^{\Omega\to\Omega} f$ is not comparable with f for some γ. Consider the function f defined by:

$$f(\xi) = \begin{cases} \xi+1, & \xi < \omega \\ \omega, & \xi = \omega, \xi = \omega+1 \\ \omega+1, & \xi > \omega+1 \end{cases}$$

Then f is monotone, but $I_\omega f$ is the function that is constantly ω, so $I_\omega f$ is not comparable with f. We also recall that we showed with the functional Φ in the previous section that we cannot extend Thm. 3.7[1] to the type $\Omega^{\mathrm{mon}} \to \Omega^{\mathrm{mon}}$.

4. Hereditarily Positive Functionals

In order to establish the desired correspondence between application of iteration functionals and arithmetic at higher type, we introduce a new notion: hereditarily positive equality. However, the result that we prove (Thm. 4.8) is technically weaker than Thm. 3.7 and cannot be used to derive the latter. Nonetheless, as the new equivalence relation is just equality on the ordinals, it is sufficient for defining them. In this section, we work in the full type structure $\mathrm{Tp}(\Omega)$.

Definition The *hereditarily positive* (*h.p.*) functionals and the order $\leq^{\mathrm{hp}}$ are defined simultaneously by induction on type as follows:

- Any element of Ω or $\Omega_{\rho\to\tau}$, $\rho \neq \tau$, is h.p.; $\leq^{\mathrm{hp}}$ in either case is just $\leq$.
- If $f \in \Omega_{\tau\to\tau}$, then f is h.p. provided:
  - If $x \in \Omega_\tau$ is h.p., then fx is h.p.;
  - f is hereditarily inflationary[2]: if $x \in \Omega_\tau$ is h.p., then $x \leq^{\mathrm{hp}} fx$;
  - f is hereditarily monotone: if x, $x' \in \Omega_\tau$ are h.p. and $x \leq^{\mathrm{hp}} x'$, then $fx \leq^{\mathrm{hp}} fx'$.

  If f, $f' \in \Omega_{\tau\to\tau}$, we say $f \leq^{\mathrm{hp}} f'$ provided that for all h.p. $x \in \Omega_\tau$, $fx \leq^{\mathrm{hp}} f'x$.

We say that $f =^{\mathrm{hp}} g$ if $f \leq^{\mathrm{hp}} g$ and $g \leq^{\mathrm{hp}} f$.

[1]Actually, it is possible to extend part (1) by using the fact that for any $\mu < \alpha$, $\limsup_{\xi\to\alpha}\{f_\xi\} = \limsup_{\mu<\xi\to\alpha}\{f_\xi\}$.

[2]We use the phrase "hereditarily inflationary" instead of the more accurate but somewhat wordier "inflationary on h.p. arguments", and similarly we say "hereditarily monotone".

We stress that the h.p. functionals do *not* form a new type structure—they are a subclass of the universe of an existing one. However, the order $\leq^{\mathrm{hp}}$ itself is defined on *all* functionals, even those that are not themselves hereditarily positive. When proving facts involving the notion of hereditarily positive, we will often use induction on type—in this situation, there are two base cases: the type Ω, and all types of the form $\Omega_{\sigma\to\tau}$ with $\sigma \neq \tau$.

Lemma 4.1

1. $\leq^{\mathrm{hp}}$ *is reflexive and transitive, and therefore* $=^{\mathrm{hp}}$ *is an equivalence relation.*
2. *If* $f \leq g$, *then* $f \leq^{\mathrm{hp}} g$; *if* $f = g$, *then* $f =^{\mathrm{hp}} g$; *if* $f, f' \in \Omega_{\sigma\to\tau}$, *then* $f =^{\mathrm{hp}} f'$ *iff for all* $x \in \Omega_\sigma$, $fx =^{\mathrm{hp}} f'x$.
3. *If* $q : \zeta \to \zeta'$ *and for all* $\xi < \zeta$, $f_\xi \leq^{\mathrm{hp}} f'_{q(\xi)}$, *then* $\limsup_{\xi\to\zeta}\{f_\xi\} \leq^{\mathrm{hp}} \limsup_{\xi\to\zeta}\{f'_{q(\xi)}\}$. *In particular, if for all* $\xi < \zeta$, $f_\xi \leq^{\mathrm{hp}} f'_\xi$, *then* $\limsup\{f_\xi\} \leq^{\mathrm{hp}} \limsup\{f'_\xi\}$, *and if* $f_\xi \leq^{\mathrm{hp}} f$ *for all* ξ, *then* $\limsup_{\xi\to\zeta}\{f_\xi\} \leq^{\mathrm{hp}} f$.
4. *If* $f_\alpha \leq^{\mathrm{hp}} f_\gamma$ *for* $\alpha < \gamma < \zeta$, *then* $\limsup_{\xi\to\zeta}\{f_\xi\} =^{\mathrm{hp}} \sup_{\xi<\zeta}\{f_\xi\}$.

Proof. (1) and (2) are immediate, and (3) and (4) are proved by induction on type. We provide details for (4). Note that this is not a trivial claim, as it is an assertion about the h.p. order, not the pointwise order. The claim is true for the base cases because the two orders are the same. Suppose $f_\xi : \Omega_\tau \to \Omega_\tau$ for $\xi < \zeta$. If $\alpha < \gamma < \zeta$ and $x \in \Omega_\tau$ is h.p., then since $f_\alpha \leq^{\mathrm{hp}} f_\gamma$, we have $f_\alpha x \leq^{\mathrm{hp}} f_\gamma x$, and hence $\left(\limsup_{\xi\to\zeta}\{f_\xi\}\right) x = \limsup_{\xi\to\zeta}\{f_\xi x\} =^{\mathrm{hp}} \sup_{\xi<\zeta}\{f_\xi x\} = \left(\sup_{\xi<\zeta}\{f_\xi\}\right) x$, with the second equality following from the induction hypothesis. ∎

Lemma 4.2 *Fix any type* σ, *countable ordinal* ζ, *and* $\{f_\xi\}_{\xi<\zeta} \subseteq \Omega_\sigma$. *If there is* α *such that* f_ξ *is h.p. for all* $\xi \geq \alpha$, *then* $\limsup_{\xi\to\zeta}\{f_\xi\}$ *is h.p.*

Proof. The lemma follows from the special case $\alpha = 0$, since the lim sup of a sequence is the same as the lim sup of any tail of that sequence. The proof is by induction on σ, using Lemma 4.1. The claim is trivially true in the base cases. Suppose that $f_\xi : \Omega_\tau \to \Omega_\tau$.

- If x is h.p., then for all ξ, $f_\xi x$ is h.p., so $(\limsup\{f_\xi\})\, x = \limsup\{f_\xi x\}$ is h.p. by the induction hypothesis.
- If x is h.p., then for all ξ we have $x \leq^{\mathrm{hp}} f_\xi x$, so $x \leq^{\mathrm{hp}} \limsup\{f_\xi x\} = (\limsup\{f_\xi\})\, x$.
- If $x \leq^{\mathrm{hp}} x'$ are h.p., then for all ξ we have $f_\xi x \leq^{\mathrm{hp}} f_\xi x'$, so $(\limsup\{f_\xi\})\, x = \limsup\{f_\xi x\} \leq^{\mathrm{hp}} \limsup\{f_\xi x'\} = (\limsup\{f_\xi\})\, x'$. ∎

As I_0 is the functional that is constantly the identity, it is not inflationary and hence not h.p. However, this is the only way in which the iteration functionals are not well-behaved: I_α is h.p. for all $\alpha \geq 1$, and the functionals I_α form a non-decreasing sequence with respect to $\leq^{\mathrm{hp}}$.

Proposition 4.3 *For each type* σ *and countable* $\alpha \geq 1$, I^σ_α *is h.p.*

Proof. The proof is by induction on α. If $\alpha = 1$, then I_α is the identity function, which is easily seen to be h.p.

Suppose that $\alpha = \gamma + 1$. First we must show that if f is h.p., then so is $I_\alpha f$, using the fact that $I_\gamma f$ is h.p. by the induction hypothesis.

- If x is h.p., then $I_\alpha f x = f(I_\gamma f x)$ is h.p. because $I_\gamma f x$ is h.p. by the induction hypothesis and f maps h.p. functionals to h.p. functionals by assumption.
- If x is h.p., then $x \leq^{\mathrm{hp}} I_\gamma f x$, so $x \leq^{\mathrm{hp}} fx \leq^{\mathrm{hp}} f(I_\gamma f x) = I_\alpha f x$. The first inequality follows from the fact that f is hereditarily inflationary, the second from the fact that f is hereditarily monotone.
- If $x \leq^{\mathrm{hp}} x'$ are h.p., then $I_\alpha f x = f(I_\gamma f x) \leq^{\mathrm{hp}} f(I_\gamma f x') = I_\alpha f x'$. The second inequality follows from the facts that $I_\gamma f$ is h.p. and f is hereditarily monotone.

To show that I_α is hereditarily inflationary, it suffices to show that if f and x are h.p., then $fx \leq^{\mathrm{hp}} I_\alpha fx$, which we did above. To show that I_α is hereditarily monotone, fix $f \leq^{\mathrm{hp}} f'$ and $x \leq^{\mathrm{hp}} x'$ and note that $I_\alpha fx = f(I_\gamma fx) \leq^{\mathrm{hp}} f(I_\gamma fx') \leq^{\mathrm{hp}} f(I_\gamma f'x') \leq^{\mathrm{hp}} f'(I_\gamma f'x') = I_\alpha f'x'$, repeatedly using the induction hypothesis and hereditary monotonicity of h.p. functionals.

If α is a limit, then $I_\alpha = \limsup_{\xi\to\alpha}\{I_\xi\}$ is h.p. by Lemma 4.2 because I_ξ is h.p. for all $1 \leq \xi < \alpha$ by the inductive hypothesis. ■

PROPOSITION 4.4 *For all countable α and γ, if $\alpha < \gamma$, then $I_\alpha \leq^{\mathrm{hp}} I_\gamma$.*

Proof. The proposition is proved by induction on γ for all $\alpha < \gamma$. Note that it is true when $\alpha = 0$, even though I_0 is not itself hereditarily positive. If $\gamma = 0$, then the claim is vacuously true.

Suppose that $\gamma = \delta + 1$ and fix any $\alpha < \gamma$. By the induction hypothesis $I_\alpha \leq^{\mathrm{hp}} I_\delta$, so it suffices to show that $I_\delta \leq^{\mathrm{hp}} I_\gamma$. To do so, fix h.p. functionals f and x. Since f and $I_\delta fx$ are h.p. (notice that this is true even when $\delta = 0$, since then $I_\delta fx = x$), $I_\delta fx \leq^{\mathrm{hp}} f(I_\delta fx) = I_\gamma fx$. Since f and x were chosen arbitrarily, $I_\delta \leq^{\mathrm{hp}} I_\gamma$.

Suppose that γ is a limit and fix any $\alpha < \gamma$. By the induction hypothesis the sequence $\{I_\xi\}_{\xi<\gamma}$ is non-decreasing with respect to $\leq^{\mathrm{hp}}$. Thus, by Lemma 4.1(4), $I_\gamma = \limsup_{\xi\to\gamma}\{I_\xi\} =^{\mathrm{hp}} \sup_{\xi<\gamma}\{I_\xi\}$. Since $\alpha < \gamma$, there is some $\delta < \gamma$ such that $\alpha < \delta$, which, by the induction hypothesis applied to δ, implies that $I_\alpha \leq^{\mathrm{hp}} I_\delta \leq^{\mathrm{hp}} \sup_{\xi<\gamma}\{I_\xi\} =^{\mathrm{hp}} I_\gamma$. ■

At this point, we are almost done, because if $\{\alpha_\xi\}_{\xi<\zeta}$ is an increasing sequence of ordinals, then $\limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\} = \sup_{\xi<\zeta}\{I_{\alpha_\xi}\}$ (recall that the difficulty was evaluating the lim sup over a subsequence). But first we need to ensure that the supremum is itself an iteration functional. With a little extra effort, we can prove a more general result: $\limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\}$ is an iteration functional for *any* sequence of ordinals $\{\alpha_\xi\}_{\xi<\zeta}$. To prove this, we combine Prop. 4.4 with the fact that the lim sup of a sequence of ordinals can always be calculated as the supremum over some tail of the sequence.

LEMMA 4.5 *For any sequence of ordinals $\{\alpha_\xi\}_{\xi<\zeta}$, there is an ordinal $\mu < \zeta$ such that* $\limsup_{\xi\to\zeta}\{\alpha_\xi\} = \sup_{\mu\leq\xi<\zeta}\{\alpha_\xi\}$.

Proof. By definition, $\limsup_{\xi\to\zeta}\{\alpha_\xi\} = \inf_{\gamma<\zeta}\{\sup_{\gamma\leq\xi<\zeta}\{\alpha_\xi\}\}$. Since any set of ordinals attains its infimum, there is some $\mu < \zeta$ such that $\inf_{\gamma<\zeta}\{\sup_{\gamma\leq\xi<\zeta}\{\alpha_\xi\}\} = \sup_{\mu\leq\xi<\zeta}\{\alpha_\xi\}$. ■

An analogous fact holds for sequences of iteration functionals:

LEMMA 4.6 *For any sequence of ordinals $\{\alpha_\xi\}_{\xi<\zeta}$, take μ as in Lemma 4.5; then* $\limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\} =^{\mathrm{hp}} \sup_{\mu\leq\xi<\zeta}\{I_{\alpha_\xi}\}$.

Proof. By the choice of μ, we have $\limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\} = \inf_{\gamma<\zeta}\{\sup_{\gamma\leq\xi<\zeta}\{I_{\alpha_\xi}\}\} \leq \sup_{\mu\leq\xi<\zeta}\{I_{\alpha_\xi}\}$. For the reverse inequality, fix any δ such that $\mu \leq \delta < \zeta$; then $\alpha_\delta \leq \sup_{\mu\leq\xi<\zeta}\{\alpha_\xi\} = \inf_{\gamma<\zeta}\{\sup_{\gamma\leq\xi<\zeta}\{\alpha_\xi\}\}$ by the choice of μ. So for any $\gamma < \zeta$, $\alpha_\delta \leq \sup_{\gamma\leq\xi<\zeta}\{\alpha_\xi\}$, and therefore there is some $\xi_\gamma \geq \gamma$ such that $\alpha_\delta \leq \alpha_{\xi_\gamma}$, which by Prop. 4.4 implies that $I_{\alpha_\delta} \leq^{\mathrm{hp}} I_{\alpha_{\xi_\gamma}}$. Keeping in mind that δ is fixed while γ was chosen arbitrarily, $I_{\alpha_\delta} \leq^{\mathrm{hp}} \limsup_{\gamma\to\zeta}\{I_{\alpha_{\xi_\gamma}}\} \leq^{\mathrm{hp}} \limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\}$; the final inequality follows from Lemma 4.1(3). Since δ was chosen arbitrarily between μ and ζ, this implies that $\sup_{\mu\leq\xi<\zeta}\{I_{\alpha_\xi}\} \leq^{\mathrm{hp}} \limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\}$. ■

PROPOSITION 4.7 *For any sequence of ordinals $\{\alpha_\xi\}_{\xi<\zeta}$, $\limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\} =^{\mathrm{hp}} I_{\limsup\{\alpha_\xi\}}$.*

Proof. Fix μ as in Lemma 4.5 and set $\alpha =_{\mathrm{df}} \limsup\{\alpha_\xi\} = \sup_{\mu\leq\xi<\zeta}\{\alpha_\xi\}$. First, suppose that for all γ there is $\xi_\gamma \geq \gamma$ such that $\alpha_{\xi_\gamma} = \alpha$. Then since $\alpha_\xi \leq \alpha$ for all $\mu \leq \xi < \zeta$, $\sup_{\mu\leq\xi<\zeta}\{I_{\alpha_\xi}\} \leq^{\mathrm{hp}} I_\alpha$. On the other hand, $\mu \leq \xi_\mu < \zeta$ and $\alpha_{\xi_\mu} = \alpha$, so $I_\alpha \leq^{\mathrm{hp}} \sup_{\mu\leq\xi<\zeta}\{I_{\alpha_\xi}\}$, and therefore $\limsup\{I_{\alpha_\xi}\} =^{\mathrm{hp}} \sup_{\mu\leq\xi<\zeta}\{I_{\alpha_\xi}\} =^{\mathrm{hp}} I_\alpha$.

If there is some γ such that $\alpha_\xi < \alpha$ for all $\xi \geq \gamma$, then we can still conclude that $\limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\} \leq^{\mathrm{hp}} I_\alpha$. Note that in this situation, α must be a limit. To show that the reverse inequality holds, fix any $\gamma < \alpha$; then there is $\delta \geq \mu$ such that $\gamma \leq \alpha_\delta$, so by Prop. 4.4 $I_\gamma \leq^{\mathrm{hp}} I_{\alpha_\delta} \leq^{\mathrm{hp}} \sup_{\mu\leq\xi<\zeta}\{I_{\alpha_\xi}\} =^{\mathrm{hp}} \limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\}$. But since γ was chosen arbitrarily, this implies that $I_\alpha = \limsup_{\gamma\to\alpha}\{I_\gamma\} \leq^{\mathrm{hp}} \limsup_{\xi\to\zeta}\{I_{\alpha_\xi}\}$. The inequality follows from Lemma 4.1(3) by considering $\limsup\{I_{\alpha_\xi}\}$ as a single h.p. functional bounding *all* of the I_γ. ■

Now we arrive at the main result relating compositions of functionals of the form I_α to ordinal arithmetic:

<u>THEOREM 4.8</u> (Iteration Functionals under $=^{\mathrm{hp}}$) *Let $f \in \Omega_{\tau\to\tau}$ and $x \in \Omega_\tau$ be h.p. Then for any countable α and γ:*

1. $I^\tau_\alpha f(I^\tau_\gamma fx) =^{\mathrm{hp}} I^\tau_{\gamma+\alpha} fx$.
2. $I^\tau_\alpha(I^\tau_\gamma f) =^{\mathrm{hp}} I^\tau_{\gamma\alpha} f$.
3. $I^{\tau\to\tau}_\alpha(I^\tau_\gamma) =^{\mathrm{hp}} I^\tau_{\gamma\alpha}$.

Proof. Each part is proved by induction on α; we do (2) as an example. If $\alpha = 0$ and x is h.p., then $I_\alpha(I_\gamma f)x = x = I_{\gamma\alpha} fx$.

If $\alpha = \delta+1$, then $I_\alpha(I_\gamma f)x = (I_\gamma f)\big(I_\delta(I_\gamma f)x\big) =^{\mathrm{hp}} (I_\gamma f)\big(I_{\gamma\delta} fx\big) =^{\mathrm{hp}} (I_{\gamma\delta+\gamma} fx) =^{\mathrm{hp}} (I_{\gamma(\delta+1)} fx)$. The second equality is the induction hypothesis and the third is an application of (1).

If α is a limit, then $I_\alpha(I_\gamma f) =^{\mathrm{hp}} \limsup_{\xi\to\alpha}\{I_\xi(I_\gamma f)\} =^{\mathrm{hp}} \limsup_{\xi\to\alpha}\{I_{\gamma\xi} f\} =^{\mathrm{hp}} I_{\gamma\alpha} f$, with the middle equality following from the induction hypothesis and the last one by Prop. 4.7. ■

It is useful to note why f and x are required to be h.p. in Theorem 4.8. In the last equality of the limit case, we use Prop. 4.7, which asserts only that $\limsup_{\xi\to\alpha}\{I_{\gamma\xi}\} =^{\mathrm{hp}} I_{\gamma\alpha}$. Thus, *when f is h.p.*, we can conclude that

$$\limsup_{\xi\to\alpha}\{I_{\gamma\xi} f\} = \Big(\limsup_{\xi\to\alpha}\{I_{\gamma\xi}\}\Big)(f) =^{\mathrm{hp}} I_{\gamma\alpha} f$$

In particular, the "alternating" function which we considered in Section 2 is not itself h.p., and this last argument would fail for that function.

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